



An answer to a question of Herings et al.

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Abstract

One answers to an open question of Herings et al. (2008), by proving that their fixed point theorem for discontinuous functions works for mappings defined on convex compact subset of \mathbf{R}^n , and not only polytopes. This rests on a fixed point result of Toussaint (1984).

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1 The result

In [2], Herings et al. prove the following new fixed point theorem for possibly discontinuous mappings:

Theorem 1.1 *Let P a non empty polytope, i.e. the convex hull of a finite subset of \mathbf{R}^n ; let $f : P \rightarrow P$ which is "locally gross direction preserving" in the following sense: for every $x \in P$ such that $f(x) \neq x$, there exists V_x , an open neighborhood of x in P such that for every u and v in V_x , $\langle f(u) - u, f(v) - v \rangle \geq 0$.*

Then f admits a fixed-point, i.e. there exists $\bar{x} \in P$ such that $f(\bar{x}) = \bar{x}$.

This theorem is a generalization of Brouwer fixed point theorem (see [1]) which says that every continuous mapping from the unit closed ball of \mathbf{R}^n to itself admits a fixed point. Yet, there is a restriction in Theorem 1.1: the set P must be a polytope. But typical strategy sets in game theory are rather compact and convex sets. Thus, an important question in practice is to know if Theorem 1.1 holds true for such subset of \mathbf{R}^n . In [3], one can read: "whether locally gross direction preserving is sufficient to guarantee the existence of a fixed point on an arbitrary non empty convex and compact set is still an open question".

In this note, we prove that Herings et al. result holds true when P is a nonempty convex and compact subset of \mathbf{R}^n . This proof rests on a fixed point theorem of S. Toussaint ([4]) we describe at the end of this paper.

Here, and throughout this paper, for every $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$, $\langle x, y \rangle$ denotes the eulidean scalar product of x and y . The set P is a nonempty convex and compact subset of \mathbf{R}^n .

Lemma 1.2 *If $f : P \rightarrow P$ is locally gross direction preserving and has no fixed points, then for every $x \in P$ such that $x \neq f(x)$, there exists $p_x \in \mathbf{R}^n$ and V_x , an open neighborhood of x in P , such that for every $x' \in V_x$, one has*

$$\langle p_x, f(x') - x' \rangle > 0.$$

Proof. Suppose that $f : P \rightarrow P$ is locally gross direction preserving and has no fixed points. Let $x \in P$, and let V_x be an open neighborhood of x in P such that for every u and v in V_x , $\langle f(u) - u, f(v) - v \rangle \geq 0$. Let $\{f(x_1) - x_1, \dots, f(x_k) - x_k\}$ be a basis of the vector space $F := \text{span}\{f(y) - y, y \in V_x\}$, where $k \in \mathbf{N}^*$, and x_1, \dots, x_k are in V_x . Then define $p_x = \sum_{i=1}^k (f(x_i) - x_i)$.

Let $x' \in V_x$. One clearly have

$$\langle p_x, f(x') - x' \rangle \geq 0 \tag{1}$$

from "locally gross direction preserving" property and from the definition of p_x . Besides, since one has $\langle f(x_i) - x_i, f(x') - x' \rangle \geq 0$ for every $i = 1, \dots, k$, Inequation 1 is an equality if and only if for every $i = 1, \dots, k$, one has $\langle f(x') - x', f(x_i) - x_i \rangle = 0$. This last property would imply $f(x') - x' \in F^\perp \cap F = \{0\}$, a contradiction with the assumption that $f(x') \neq x'$. Thus, Inequality 1 is strict, which ends the proof of the lemma.

Now, for every $x \in P$, one introduces the (possibly empty) multivalued mapping T_x defined for every $y \in P$ by

$$T_x(y) = \{z \in P, \langle z - y, p_x \rangle > 0\}.$$

Clearly, for every $x \in P$, T_x has convex values. Besides, for every $y \in P$, one has $y \notin T_x(y)$. Lastly, for every $z \in P$, the set $T_x^{-1}(z) = \{y \in P \mid z \in T_x(y)\}$ is an open subset of P . A multivalued mapping T_x satisfying theses assumptions is called KF in [4].

Besides, from the definition of p_x , there exists a neighborhood V_x of x such that for every $x' \in V_x$, one has $f(x') \in T_x(x')$: one says that T_x is a KF -majorant of f at x (see [4]).

To summarize, we have proved that a locally gross direction preserving mapping $f : P \rightarrow P$ without fixed point is KF -majorized in the sense that for every $x \in P$, there exists a KF -majorant T_x of f at x .

Now, state Theorem 2.2. in [4]: it says that given any non empty convex and compact subset X of a topological space, given any multivalued mapping $F : X \rightarrow X$ which is KF -majorized, there exists $\bar{x} \in X$ such that $F(\bar{x}) = \emptyset$. Apply this theorem to $F = f$ and $X = P$. Thus, there should exists $\bar{x} \in P$ such that $f(\bar{x}) = \emptyset$, a contradiction. This proves that any locally gross direction preserving mapping $f : P \rightarrow P$ must admit a fixed point.

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References

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